

A NOISE THEORY FOR THE MAGNETRON
I. THE TEMPERATURE LIMITED LOW CURRENT
DENSITY MAGNETRON

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A NOISE THEORY FOR THE MAGNETRON

I. The Temperature Limited Low Current Density Magnetron

R. Q. Twiss

This report and related Technical Reports No. 117 and 118 present the material of a doctoral Thesis in the Department of Electrical Engineering, M.I.T.

Abstract

In Part I of this paper a noise theory is given for a temperature limited magnetron, valid for sufficiently low current densities, from which one can calculate the shot noise produced by such a tube in terms of the magnetron parameters and of the orbit of an individual electron. This theoretical noise output is compared with that actually produced by an experimental c-w magnetron type QK61 and two important differences are noted. The observed noise is as much as 30 db above the theoretical shot noise under certain conditions, while the observed rate of change of noise power with plate voltage is many times greater than that predicted for the theoretical shot noise. A general discussion of the possible origins of this excess noise is given and a theory to explain it is given in Part II of this paper, Technical Reports Nos. 117 and 118.

LIST OF SYMBOLS

A	cathode area
c	velocity of light $2.997 \cdot 10^8$ m/sec
d	width of magnetron slot
e	electronic charge $1.6 \cdot 10^{-19}$ coulomb
h	axial length of magnetron
I(t)	current density at time t
I _c	cathode emission current density
I _a	anode current density
k	Boltzmann's constant $1.37 \cdot 10^{-23}$ joule/degree K
L	length of anode slot
m	parameter of mode
m _o	electronic mass $9.11 \cdot 10^{-31}$ kg
N	number of anode slots
Q _m	loaded magnetron Q
Q _L	external Q
Q _{mo}	magnetron unloaded Q in neighborhood of m th resonance
r _c	cathode radius
r _a	anode radius
R _t	resistive termination to slot
T _r	room temperature 290°C
Y _{mc} Y _{ms}	orbit parameter. See Eq. 51
Z _o	impedance of free space, 377 ohms
α	reflection coefficient or parameter
ε _o	dielectric constant of free space $1/36\pi \cdot 10^9$
λ	emission probability parameter
μ _o	magnetic permeability at free space $4\pi \cdot 10^{-7}$
ν	frequency
ν _{mo}	resonant frequency associated with m th mode
τ	transit time
ω	angular frequency
ω _{mo}	resonant angular frequency associated with m th mode

A NOISE THEORY FOR THE TEMPERATURE LIMITED LOW CURRENT DENSITY MAGNETRON

Introduction

Since the discovery of the resonant anode magnetron, at the beginning of World War II, a great deal of time and energy has been devoted to the design and theory of such tubes for use as oscillators in the centimeter waveband. Despite this, there are still a number of striking but unexplained anomalies in the behavior of the non-oscillating magnetron operated with constant plate potential and magnetic field.

The most significant of these was reported by E. G. Linder (1) who obtained experimental curves showing the variation of plate current with plate voltage and magnetic field in a single anode magnetron. The rate of decrease of anode current with increasing magnetic field differed by many orders of magnitude from that predicted by theory and corresponded to initial electron energies of between 10^2 and 10^3 electron volts. If this interpretation were taken literally it would imply cathode temperatures of 10^6 to 10^7 °C.

Only less extraordinary were some measurements, reported by F. F. Rieke (2), on the noise properties of non-oscillating magnetrons operated with fixed plate potential. He concluded that, with typical values of magnetic field, the fluctuation noise power produced in the magnetron load was more than 40 db in excess of that attributable to shot noise alone.

In the first part of this paper we consider only the second of these two phenomena. After a general discussion of the possible origins of excess noise in a tube, a quantitative analysis is given for the shot noise power produced by a temperature limited magnetron.

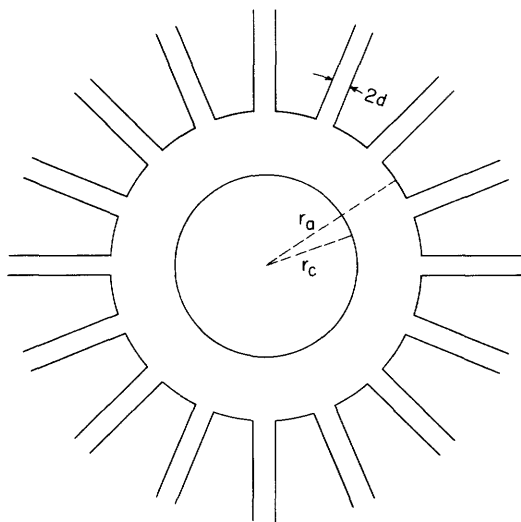


Fig. 1 16-slot cylindrical magnetron

$$r_c = 1.75 \text{ mm}$$

$$r_a = 2.85 \text{ mm}$$

$$d = 0.30 \text{ mm}$$

$$h, \text{ axial length} = 1.0 \text{ cm.}$$

By shot noise power we mean the noise output power of a tube calculated on the assumption that interaction between the individual electrons of the space charge can be ignored as far as the fluctuations are concerned and that this interaction only affects the steady state conditions. A comparison with experiment shows that this shot noise is about 30 db less than that observed under certain conditions. This discrepancy is less than that observed by Rieke but is still a very large quantity. The experimental results given in this paper were obtained with a cylindrical 16-slot magnetron, details of which are shown in Fig. 1.

In Part II of this paper (Technical Reports No. 117 and 118) an attempt is made to develop a theory to explain this excess noise and a

tentative explanation is also given for the excess plate current observed by Linder. Reasonable agreement is obtained with the experimental evidence available to date but a considerable amount of further data must be obtained before the basic hypotheses of this theory can be regarded as established.

I. CAMPBELL'S THEOREM AND NOISE REPRESENTATIONS

A. Noise Representations. In the past, there has been a great deal of controversy about the correct theoretical approach to the problems of random fluctuations; the chief point of dispute is the validity of the Fourier series and Fourier integral representations of noise currents which have been employed by the majority of workers in this field. The most detailed attack on the Fourier representation was delivered by N.R. Campbell and J. V. Francis (5), who suggested that rigor can only be obtained by confining attention to the fluctuations of the instrument which is being used to measure the noise currents, the so-called "shot-noise" representation. It is now universally recognized, however, that the objections voiced by Campbell to the Fourier representation can be overcome and that a perfectly rigorous analysis based thereon can be set up (6). In fact the two representations are, mathematically, fully equivalent and either one can be used as the basis for a complete theory (7).

However, when we come to analyze the fluctuations that exist in a concrete physical problem, it may well be that one representation has appreciable advantages over the other from the point of view of physical understanding of the phenomenon. Thus the Fourier representation is peculiarly well adapted for finding the fluctuations of an assembly in thermal equilibrium with its surroundings, while the "shot-effect" representation is often the better when the dynamical trajectory of every particle can be calculated.

In what follows we shall feel free to use either representation but the majority of our analysis is based upon the shot representation and Campbell's theorem (8). A full discussion of the derivation of this theorem, and of its use in analyzing the fluctuating currents in conventional amplifier tubes is given in the paper (5) already referred to.

B. Campbell's Theorem. The term noise current, as used in this report, is applied to those tube current fluctuating components which arise because of the random emission from the cathode of the electrons that compose the current. The output noise power of a tube, which is the quantity experimentally measured, is proportional to the mean square value of these fluctuations. The basic theorem of tube noise, which is Campbell's theorem, enables us to calculate this noise power.

Suppose that the emission of an electron charge, e , from the cathode at time $t = t_k$ produces an effect of $ef(t - t_k)$ on a measuring instrument in the output circuit. This measuring instrument might be an ammeter in the tube anode-cathode circuit or an ammeter in the output of an amplifier connected to the tube. If the measuring instrument

be such that the effect of the various electrons add linearly, then the total effect at time t due to all the electrons is

$$I(t) = \sum_{k=-\infty}^{\infty} e f(t - t_k) . \quad (1)$$

Then Campbell's theorem states that the average value of $I(t)$ is

$$\overline{I(t)} = \lambda e \int_{-\infty}^{\infty} f(t) dt \quad (2)$$

and the mean square average of the fluctuation about this value is

$$\overline{[I(t) - \overline{I(t)}]^2} = \lambda e^2 \int_{-\infty}^{\infty} f^2(t) dt \quad (3)$$

where λdt is the probability that an electron be emitted in time dt . We see that λ is the average number of electrons emitted per second, while λe is equal to I_c the average cathode current. As it stands, Campbell's theorem is applicable to the case where there are initial events of one kind only, that is to say, to the case where all electrons produce the same effect and it relates the mean square fluctuations in the reading of the measuring instrument to the effect, on the instrument, of the emission of unit charge at time $t = 0$.

A considerable number of extensions of various kinds have been made to Campbell's theorem so that it can be applied to cases not covered by the restrictions outlined above. In this paper we shall need to apply Campbell's theorem to the case when there are an infinite number of distinct and independent primary events.

Let $\alpha, \beta \dots \gamma$ be a number of parameters each capable of assuming a continuum of values. Let $\lambda(\alpha, \beta \dots \gamma) d\alpha d\beta \dots d\gamma$ be the probability that an event be initiated in time dt which produces an effect $e f(t)_{\alpha, \beta \dots \gamma}$ on some measuring instrument. Then

$$\overline{V(t)} = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \dots \int_{\gamma_1}^{\gamma_2} e \lambda(\alpha, \beta \dots \gamma) d\alpha d\beta \dots d\gamma \int_{-\infty}^{\infty} f(t)_{\alpha, \beta \dots \gamma} dt \quad (4)$$

and

$$\overline{[V(t) - \overline{V(t)}]^2} = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \dots \int_{\gamma_1}^{\gamma_2} e^2 \lambda(\alpha, \beta \dots \gamma) d\alpha d\beta \dots d\gamma \int_{-\infty}^{\infty} f^2(t)_{\alpha, \beta \dots \gamma} dt . \quad (5)$$

The proof of this extension to the continuous case presents no difficulties if all the integrals can be defined in the sense of Riemann, and are uniformly continuous. This will indeed be the case in every application that we shall make of this theorem.

In the case of the magnetron the effect of an electron emitted from the cathode depends

upon its initial normal and tangential velocities, which are distributed according to a Maxwell-Boltzmann law, and upon the angular coordinate of the point of emission. To find the total noise fluctuations it is necessary to average these three parameters.

C. Campbell's Theorem and Parseval's Theorem. Campbell's theorem, so stated above, deals solely with events in the time domain and, in any direct experimental verification, it would therefore be necessary to measure the transient response of the measuring instrument to a single event. This, in practice, is usually very difficult if not impossible. However, we shall show, that to find the mean square fluctuation of the measuring instrument, it is sufficient to know the absolute magnitude of the response of the measuring instrument to a signal of constant amplitude and constant but arbitrary frequency; a much more easily measurable property. This equivalence can easily be established as a consequence of Parseval's theorem for Fourier integrals (9). This theorem can be stated as follows:

Let $F(i\omega)$ be the Fourier transform of the L-2 integrable function of $f(t)$ so that

$$F(i\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) \exp(i\omega t) d\omega \quad (6)$$

then

$$\int_{-\infty}^{\infty} f^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F^2(i\omega)| d\omega \quad (7)$$

If we introduce the frequency $\nu = \omega/2\pi$ in place of the angular frequency ω and if we remember that

$$|F^2(i\omega)| = F(i\omega) F^*(i\omega) = F^*(-i\omega) F(-i\omega) = |F^2(-i\omega)| \quad (8)$$

we can write this result in the alternative form

$$\int_{-\infty}^{\infty} f^2(t) dt = 2 \int_0^{\infty} |F^2(\nu)| d\nu \quad (9)$$

With the aid of this identity we can write Campbell's theorem in the form

$$\overline{[V(t) - \overline{V(t)}]^2} = 2e^2 \int_{a_1}^{a_2} \lambda(a) da \int_0^{\infty} |F_a^2(\nu)| d\nu \quad (10)$$

where

$$F_a(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i \nu t) dt$$

and where, for simplicity, we have assumed that there is only one parameter distinguishing the fundamental events.

It should be pointed out that the use of Campbell's theorem in this equivalent form does not mean that we have abandoned the use of a shot effect representation in favor of a Fourier representation. Such a step would require us to provide a Fourier analysis of the fluctuating readings of the measuring instrument, while we have simply introduced the Fourier analysis of the response of the measuring instrument to an event at time $t = 0$.

Another advantage of the alternative form of Campbell's theorem, is that it readily lends itself to the erection of equivalent noise representations which are very useful in comparing the noise properties of various tubes.

II. GENERAL DISCUSSION ON POSSIBLE ORIGINS OF NOISE

A. Shot Effect, Secondary Emission and Gas Scattering. If one turns to a book or journal article devoted to the subject of noise in tubes one is likely to find reference there to many kinds of noise. Shot noise, partition noise, induced grid noise being three of the more common. These distinctions should not bind us to the fact that the basic origin of the fluctuations lies simply in the random occurrence of the fundamental event, which is the emission of the electron from the cathode. If electrons were emitted with perfect regularity from the cathode there would be no shot noise, no partition noise and no induced grid noise, except at frequencies commensurable with the inverse of the time interval between successive electrons. There are of course other possible origins of noise. Thus if the electrons emitted from the cathode were permitted to fall upon a secondary emitting surface, noise currents would arise even if the initial electron flow were perfectly regular, because a given electron would produce a randomly distributed number of secondaries. Similarly, if the cathode emitted electron beam were to interact with a molecular gas in thermal equilibrium, the individual electrons would be randomly scattered by the molecules, and the current contained in a fixed solid angle would exhibit random fluctuations.

In the magnetron case, however, the initial current is randomly emitted, secondary emission is unimportant and the effects of gas scattering will be small (see page 11), as this can neither increase nor diminish the randomness of the initial stream.

Accordingly, when looking for an explanation of the noise properties of a tube we do not seek for a new origin of noise but consider instead the effect produced by an individual electron and the ways by which this can be modified in a particular case.

In the magnetron the chief problem is the enormous noise power produced. We shall therefore give a brief survey of the possible causes of excess noise in tubes with particular reference to those which are likely to be important in the magnetron. Before doing this it might be helpful to look a little more closely at Campbell's theorem which is to be used as a basis for this discussion.

B. Campbell's Theorem and its Verification. In the simplest case, when there are events of only one kind*, Campbell's theorem, as given in Eq. 3, states that the mean square fluctuations of the measuring instrument are given by

$$\overline{[V(t) - \overline{V(t)}]^2} = e^2 \lambda \int_{-\infty}^{\infty} f^2(t) dt = e I_c \int_{-\infty}^{\infty} f^2(t) dt \quad (11)$$

where $V(t)$ is the reading of the measuring instrument at time t , I_c is the average emitted current and $f(t)$ is the effect produced on the measuring instrument by unit charge emitted at time $t = 0$.

As stated in Section I, the measuring instrument might well be an ammeter or voltmeter connected in the output circuit of an amplifier following the tube. We can imagine that this output current or voltage is displayed on the screen of a cathode ray oscillograph. Ideally it would be possible to take a continuous series of photographs of the screen and so obtain a continuous record of the output of the amplifier. From this we could obtain the mean square fluctuations of the output and, if we wished, the higher moments of the output reading: the correlation function and so forth. If we could measure or calculate the response of the cathode ray tube to the emission of unit charge from the cathode of the original tube it would be possible, in this manner, to verify Campbell's theorem from the measured noise output. This last is, however, often very difficult to do. In addition practical noise measurements are not performed in this way when only the mean square fluctuations are of interest. Instead the output of the amplifier is connected to a square law detector with a long time constant in the video output. The average or steady state value of this output is then proportional to the mean square voltage fluctuations across the detector.

Now that we are no longer measuring the time distribution of the fluctuations it is much better to go over to the alternative form of Campbell's theorem as given in Eq. 10 which states, in the present simple case, that

$$\overline{[V(t) - \overline{V(t)}]^2} = 2e I_c \int_0^{\infty} |F^2(\nu)| d\nu \quad (12)$$

where

$$F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i \nu t) dt \quad (13)$$

ν is the frequency and $F(\nu)$ is the Fourier transform of $f(t)$.

Now in general it is the mean square fluctuating noise power produced by the tube in which we are interested rather than in the output noise of the amplifier. Accordingly

* This applies to the case where the effect produced by an electron does not depend appreciably on its initial velocity or on the coordinates of the point on the cathode from which it is emitted.

let $I(t)$ be the current flowing in the load resistor R in the tube anode circuit. Then

$$\overline{[I(t) - \overline{I(t)}]^2} R$$

is the noise power delivered to the load. If $g(t)$ is the current produced in the load by emission of a single electron at time $t = 0$ and if $G(\nu)$ is the Fourier transform of $g(t)$ we have, e.g.,

$$\overline{[I(t) - \overline{I(t)}]^2} R = 2e I_c R \int_0^\infty |G^2(\nu)| d\nu = R \int d\overline{i^2(\nu)} \quad (14)$$

where

$$d\overline{i^2(\nu)} = 2e I_c |G^2(\nu)| d\nu \quad (15)$$

$R d\overline{i^2(\nu)}$ is the noise power dissipated in the load in bandwidth $d\nu$. Now by definition,

$$A^2(\nu) d\overline{i^2(\nu)} = |F^2(\nu)| d\nu \quad (16)$$

where $A(\nu)$ is the amplitude response of the amplifier.

In the most common practical case the amplifier is designed so that $A(\nu)$ is approximately constant and equal to $A(\nu_0)$ over a band

$$\nu_0 < \nu < \nu_0 + \Delta\nu$$

and is approximately zero outside this band. $\Delta\nu$ is a bandwidth so small that the variation of $d\overline{i^2(\nu)}$ in this range is negligible. In this special but important case, the mean square fluctuations at the output of the amplifier are given by

$$\begin{aligned} \overline{[V(t) - \overline{V(t)}]^2} &= 2e I_c \int_0^\infty |F^2(\nu)| d\nu \\ &= 2e I_c |F^2(\nu)| \Delta\nu \end{aligned}$$

while

$$\begin{aligned} d\overline{i^2(\nu)} &= \frac{F^2(\nu)}{A^2(\nu_0)} \Delta\nu \frac{d\nu}{\Delta\nu} 2e I_c \\ &= \frac{\overline{[V(t) - \overline{V(t)}]^2}}{A^2(\nu_0)} \frac{d\nu}{\Delta\nu} \quad (17) \end{aligned}$$

All the quantities on the right hand side of Eq. 17 are measurable so that $R d\overline{i^2(\nu)} = P(\nu_0) d\nu$ can be found. This is the quantity in which we are primarily concerned in this paper.

C. Possible Causes of Excess Noise in Tubes. From Eq. 15 we see that the output noise power of the tube in bandwidth $d\nu$ is given by

$$P(\nu_0) d\nu = R d\overline{i^2(\nu)} = 2e I_c R |G^2(\nu)| d\nu \quad .$$

If we neglect the effect of increasing R there are just three ways by which this quantity can be increased: 1. by increasing e , 2. by increasing I_c , 3. by increasing $|G^2(\nu)|$. The most important of these alternatives is the last but we shall consider them briefly in order before dwelling at some length on 3.

1. An actual increase in e could only be achieved by using n multipli-ionized atoms in place of electrons as the fundamental units of charge, but an effective increase in e can be obtained in an electron multiplier. Thus if each electron hitting the secondary emitting surface produces three secondaries, the electronic charge is effectively tripled. A similar effect can also be obtained in the presence of a space charge produced potential minimum, as discussed in Section IV.

2. When comparing the "noisiness" of two tubes it is quite customary to compare their noise power outputs when they are drawing the same output currents. It should be remembered however that the I_c of Eq. 12 represent the currents leaving the cathode so that, if this be large compared with the plate current, the tube may appear far noisier than one, like the temperature limited diode, where all the cathode current reaches the plate. This is particularly true in the magnetron case, where the plate current is, ideally, vanishingly small, and the circulating current that produces the fluctuating currents in the magnetron load is often very large.

3. In most cases, however, the different noise behavior of different tubes is traceable to the different effects produced by a single event rather than to either of these two causes. It is obviously impossible to discuss this question in general terms so we shall, instead, consider a number of special cases chosen to throw light on the magnetron itself.

D. Internal Noise Amplification. One obvious mechanism for obtaining increased noise is to include in the tube an amplifier with unit gain at d-c but large gain over an arbitrary frequency band. A simple illustration is afforded by a temperature limited diode followed by an a-c coupled amplifier of high gain. The fluctuations in the output current of the last tube of the amplifier will then be enormously greater than the fluctuations attributable to the shot noise of this tube at least for frequencies in the pass-band of the amplifier. This is of course an exceptionally artificial example and a number of more realistic cases can easily be found. For example there is the phenomenon known as induced grid noise.

It is found experimentally that the noise output of a triode, with an impedance connected between cathode and grid, increases steadily with frequency. This excess noise is really part of the shot noise, amplified by the tube itself, its origin being due to the random emission of electrons from the cathode and to no other source. The effect takes place as follows. An electron emitted from the cathode at time $t = 0$ traveling to the plate will induce a current in the grid cathode circuit. The induced current will produce a voltage $V(t)$ the form of which will depend upon the nature of the grid impedance. The induced voltage will react upon the average current flow and produce a current $g V(t)$

where g is the transconductance of the tube. The total effect of the emission of an electron is thus to produce a plate current equal to

$$i_a(t) = i_g(t) + gV(t) .$$

The Fourier transform of this function may be written

$$i_a(\nu) + i_g(\nu) (g Z(\nu) - 1)$$

where $i_g(\nu)$ is the Fourier transform of the induced grid current, $i_a(\nu)$ is the Fourier transform of the plate current when $Z(\nu)$ is zero, and $Z(\nu)$ is the grid-cathode impedance at frequency ν .

The fluctuations in the plate current are proportional to

$$\left[i_a(\nu) + i_g(\nu) (g Z(\nu) - 1) \right]^2$$

and can be made arbitrarily large by making $Z(\nu)$ large, thus giving an effective amplification of noise.

Exactly the same principle operates in the case of the klystron amplifier where the noise currents in the electron beam excite fluctuating voltages across the buncher cavity, which, in turn, react upon the electron beam.

A more striking example is afforded by the traveling wave tube where the interaction between the space charge and the electromagnetic waves guided by the helix provides considerable amplification of the initial fluctuation currents, gain as great as 80-100 db being possible.

Finally, and most interesting of all for our present purposes, is the so-called electron wave tube developed by A. V. Haeff (10) and his associates at the Naval Research Laboratory. This tube provides amplification of both signal and noise by the interaction of two superimposed beams of electrons and the mean velocity of one beam differs from that in the others. No external resonant or guiding circuits are required and gains of 80-100 db have been reported for this tube also. The problem of noise amplification in the magnetron space charge is discussed in Part II and it is there shown that a mechanism, similar to that operating in the electron wave tube, plays an important part in the explanation of the huge excess magnetron noise.

E. Large Noise Without Amplification. In the previous section we discussed tubes whose increased noise outputs were attributable to internal amplification. Now the amplification factor is usually itself proportional to the current flowing in the tube so that the variation of noise power with average current will contain terms proportional to I_a^2 and I_a^3 in contrast to the diode where noise power is proportional to I_a . It is, however, possible for a tube to have a very much greater noise output than the corresponding diode even when no internal amplification takes place. To see how this may be so let us consider once again the temperature limited diode.

The equivalent noise current generator (Eq. 17) for this tube is given by

$$\overline{i^2(\nu)} d\nu = 2e I_a d\nu$$

where I_a is the anode or cathode current. This expression is valid for frequencies small compared with the inverse transit time.

Now in the diode we can take, as the fundamental event, the random storage of a charge $-e$ in the anode-cathode capacity C , which is the same as storing an energy e^2/C in this capacitor. The storage, which can be regarded as instantaneous if we confine attention to low frequencies, involves an extremely small amount of energy. If C is $10 \mu\text{mf}$, a typical magnitude, then the energy stored in the capacitor is 1.6×10^{-14} electron volt. It should be noted that the quantity of energy thus stored is quite independent of the kinetic energy of the electron when it reaches the anode. This energy, which is of the order of 100 ev or 10^{16} times the stored electromagnetic energy, is entirely dissipated in heat. However, if even a small portion of this energy could be converted into electromagnetic energy the possibility of obtaining enormous noise currents, associated with a small average current, would arise.

An impressive illustration of this possibility is provided by the phenomena known as Cerenkov radiation. When a high speed electron passes through a dielectric medium in which $c\sqrt{\epsilon_0/\epsilon}$, the velocity of light propagation in the medium, is less than βc , the velocity of the electron, then radiation is observed in a direction making an angle

$$\theta = \cos^{-1} \left(\frac{1}{\beta} \sqrt{\frac{\epsilon_0}{\epsilon}} \right) \quad (19)$$

with the direction of motion of the electron. The energy radiated in this way, by a high speed electron, may well amount to several Kev/cm path length. Ideally, with the aid of mirrors, lenses and impedance matching devices, it would be possible to collect an appreciable fraction of this energy in a resonant load. If a number of electrons were fired incoherently through the medium, the mean square fluctuations produced across the load could be as much as 10^{17} times larger than the fluctuations produced in the same load in the anode circuit of a temperature limited diode drawing the same current.

As a less artificial case, let us consider a cylindrical tube consisting simply of the anode of a resonant-slot magnetron. Let electrons be injected into the device from a gun with just sufficient energy to describe circles, concentric with the cylindrical tube, under the action of a magnetic field parallel to the axis of the cylinder. The rotating electron will set up electromagnetic fields and, if one of the slots be coupled to an external load, there will be a continuous flow of energy into this load which will have to come from the kinetic energy of the electron. As the electron loses energy the radius of its orbit will continually decrease, until, finally, the electron comes to rest on the axis of the cylinder. If electrons are fired at random into the cylinder the resulting fluctuations in the load will again be 10^{17} times as large as if the load were placed in the anode circuit of a temperature limited diode, although the frequency spectrum of the output noise in the former case will not be linear. (Since this example was produced to illustrate a

general principle we have not attempted to consider such a practical question as how the electron is to escape hitting the gun on a subsequent revolution. A weak electric field parallel to the axis of the cylinder would provide a possible mechanism.)

Clearly, this last example is closely related to the case of the magnetron itself.

F. Possible Reasons for Excess Noise in Magnetrons. In the earlier portions of this section we have suggested that the fundamental origin of noise in magnetrons, as in other thermionic tubes, is simply the random emission of the individual electrons that compose the magnetron current, and we have identified three possible reasons why the magnetron noise should be so large.

1. The circulating current is much larger than the anode current and, in the magnetron, the effect produced by an electron that approaches close to the anode, but does not actually reach it, is likely to be as large if not larger than the effect of an electron which does strike the anode.

2. Owing to the long transit time of an electron in a space-charge limited magnetron there is the possibility that the effect produced by a single electron will be larger than if its sole effect was to store its charge in the anode cathode capacity.

3. As we shall show in Part II, interaction between the ingoing and outgoing streams in a magnetron provides a mechanism whereby considerable noise amplification can take place, within the space charge itself, over a wide frequency range.

As we shall see later the second of these causes is not effective in the magnetron as, under practical operating conditions, the transit time is never long enough. However, the first and third are important.

The question may well be asked as to whether there are not additional reasons for the excess magnetron noise.

One possibility, that occasionally has been mooted, is that ions may be contributing to the noise. (These ions are produced either by back bombardment on the cathode or from gas molecules present in the interaction space.) Now ions can affect the magnetron noise in two ways. Firstly by their random production and subsequent motion through the electrostatic potential minimum in front of the cathode, secondly by Rutherford scattering, which will alter the trajectory of an electron inside the magnetron. Because of their much greater mass the velocity of the ions through the electron cloud is very small compared with the electron velocity and their transit time is correspondingly long. Accordingly the fluctuations in magnetron current caused by the random production of ions will have no high frequencies present and hence cannot affect the fluctuating currents in the load of a non-oscillating magnetron. In the oscillating magnetron, however, things are different. The tube and space charge impedances are non-linear and the low frequency noise current due to the ions can beat with the oscillation frequency and spread its spectrum over a band. Since there is some reason to believe that the magnetron space charge is indeed oscillating, even when operated in a cut-off condition, we might expect an additional service of noise to arise from the presence of ions. However, this effect should

be noticeable only at frequencies near the oscillating frequency and should be negligible at frequencies far removed therefrom. The effect of Rutherford scattering is unlikely to be important for two reasons: the gas pressure in a magnetron is very low; and a random change in electron trajectory about the mean will, not to the first order, affect the mean square fluctuations.

Another possible origin of noise lies in the production of one or more secondary electrons when an ion is formed. This is also likely to be unimportant because of the low gas pressure.

These arguments against the importance of gas as a source of excess noise are not conclusive and it is desirable to obtain experimental confirmation by observing the effect, on the magnetron noise, of altering the gas pressure in the tube.

This discussion virtually exhausts the possibilities for excess noise when we confine attention to events of one kind. But in a space charge limited tube where a potential minimum exists in front of the cathode, the effect produced by an electron depends very much upon its initial velocity. It would be natural to take as the I_c of Eq. 17 the current that actually crosses the potential minimum. However, in the magnetron it appears that a considerable portion of the output noise is produced by fluctuations in the emission of electrons that do not cross the minimum and this fact is responsible for a considerable part of the excess noise.

This completes the qualitative introductory part of this paper. We shall now pass on to develop a quantitative analysis of the temperature limited magnetron with low current density.

III. THE NOISE PROPERTIES OF A TEMPERATURE LIMITED LOW CURRENT MAGNETRON

A. Introduction. The purpose of this paper is to explain the excess noise of magnetrons but we have not defined the standard by reference to which we conclude that the experimentally observed noise is excessive. A number of alternative standards have been suggested in the past. Some of them were extremely unreasonable. One suggestion was to compare the magnetron output noise with the noise delivered by a temperature limited diode, with zero transit time, drawing the same anode current. Purely from the experimental point of view this is not illogical, although the thermal noise power developed by the magnetron load is a much more absolute standard. However, it would be quite absurd to label the difference between the magnetron output and the diode output noise power, "excess" noise, since, first, the mechanisms by which noise is produced in the two tubes are quite different and, second, the plate current in a cut-off magnetron is itself an anomaly requiring explanation. In this paper we take as the standard, a temperature limited magnetron where the interaction between individual electrons is completely neglected. It will then be assumed that the orbit of an electron and the total circulating current are the same as the corresponding quantities in the magnetron whose

noise power is being measured. (This noise power may properly be called the shot noise power.) This choice of a standard is arbitrary but it is probably the most convenient for our present purposes. Accordingly we shall derive a theory for the temperature limited magnetron* which will also be of considerable help as a guide to the solution of the much more complex space charge limited case.

From Campbell's theorem (see Section I-B), the output noise power of the temperature limited magnetron can be written down once we have found the voltage $v(t)$ or, alternatively, the Fourier transform $V(i\omega)$ of $v(t)$, developed across the magnetron load by an electron emitted, at time $t = 0$, from a particular point on the cathode surface having cylindrical coordinates (r_c, θ_0) . The form of $v(t)$ will depend on θ_0 , as well as on the initial normal and tangential velocities. The latter have very little effect in the temperature limited case, only the former are important. To find the mean square fluctuations across the load we must use the form of Campbell's theorem given by Eq. 5 which is appropriate to the case where the set of fundamental events forms a continuum. By inspection of Eq. 5 we see that this is equivalent to averaging the output noise power over θ_0 . In the important special case where the emissivity of the cathode is independent of θ_0 the noise power associated with one particular mode of the cathode-anode interaction space is independent of that associated with any other mode, a fact which makes for a considerable simplification.

When an expression has been obtained for the mean square voltage fluctuations across the load it will be possible to set up an equivalent noise circuit for the magnetron and hence compare the "noisiness" of the tube with the thermal noise power in the load or, if we wish, with the noise output of a temperature limited diode.

B. Simplifying Approximations. A certain number of simplifying approximations will be made in the course of this analysis and will be stated at the stage of the theory where they are introduced. To give the reader a clear idea of the limitations of the solution, it seems best to state the more important of them explicitly at this stage, leaving an explanation of their relevance to the appropriate part of the text. It will be noted that, in general, the approximations we are making are those customarily emphasized in the mathematical theory of the magnetron (2).

1. Two-dimensional analysis. In the majority of theoretical treatments of the magnetron it is assumed that the tube is infinitely long and that all physical quantities are constant along lines parallel to the axis of the tube. We shall follow the same procedure here and, in addition, substitute for the point charge of strength e , a line charge of strength e/h per unit length, where h is the axial length of the magnetron.

*For the sake of conciseness we shall apply the term "temperature limited", in this section, to the case where the current density is so low that electron-electron interaction can be neglected. We shall not assume this limitation in Part II where we shall consider the temperature limited magnetron operated with high space charge densities.

2. Non-relativistic treatment. Throughout this report all relativistic effects are ignored, which is justified provided $eV_a \ll m_0 c^2$ where V_a is the plate-cathode potential difference. This implies that we neglect the difference between the advanced and retarded Coulomb fields, assume the velocity of the electrons very small compared with that of light, and neglect the effects of the magnetic fields produced by the electrons.

3. Magnetron design and loading. The magnetron discussed in this section has a cylindrical cathode-anode interaction space and N identical rectangular slots (Fig. 1). To avoid the introduction of asymmetry we shall assume that the magnetron slots are all equally loaded by a pure resistive termination. If, as we shall assume, the slots be tightly coupled, the total power delivered to those N loads is equal to the power delivered to a single load chosen so that the output circuit has the same bandwidth. We also assume that the magnetron walls are perfectly conducting and that all the dissipation takes place in the loads.

4. Narrow slots. The fields in the rectangular slots will be expressed in terms of orthogonal Cartesian coordinates, while the fields in the anode-cathode interaction space will be expressed in cylindrical coordinates. However, when matching fields across the boundary between the two spaces we shall assume the slots to be so narrow that the x and y field components in the slots may be matched to the r and θ field components respectively in the interaction space.

5. Matching conditions. Maxwell's equations require that both the E and the H fields be continuous across the mouth of the slots, and that the tangential components of E be zero at all metal surfaces. This requirement can only be met exactly if an infinite number of modes be set up, both in the anode-cathode interaction space and in the slots. At operating frequencies, all but one of the modes in the slot are evanescent, so that only in this, the lowest mode, can power be transferred to the load. This provides the justification for the universally adopted matching procedure that we follow in this paper. In the slot we neglect all the evanescent modes, which, by definition, carry no power. Then we require that E_e be continuous across the mouth of the slot and that B_z , the magnetic field in the interaction space, equal B_z in the slot at the center of the slot.

C. The Fields Set Up in a Magnetron by an Arbitrarily Moving Line Charge. In the M.K.S. system of units Maxwell's equations may be written

$$\begin{aligned} \text{curl } \underline{\underline{E}} &= - \frac{\partial \underline{\underline{B}}}{\partial t} & \text{curl } \underline{\underline{H}} &= \rho \underline{\underline{y}} + \frac{\partial \underline{\underline{D}}}{\partial t} \\ \text{div } \underline{\underline{B}} &= 0 & \text{div } \underline{\underline{D}} &= \rho \end{aligned} \quad (20)$$

together with the additional relations, valid in free space,

$$\begin{aligned} \underline{\underline{D}} &= \epsilon_0 \underline{\underline{E}} & \underline{\underline{B}} &= \mu_0 \underline{\underline{H}} \end{aligned}$$

where

$$\begin{aligned} \mu_0 \epsilon_0 &= 1/c^2 & \mu_0 &= 4\pi \times 10^{-7} \end{aligned} \quad (21)$$

In the present case ρ is zero everywhere except for a pole at the position of the line charge. The boundary conditions require that E_t and B_n be zero over the anode and cathode surfaces. Because of the two-dimensional nature of the problem we may take

$$\begin{aligned}\underline{E} &= (E_r, E_\theta, 0) \\ \underline{B} &= (0, 0, B_z)\end{aligned}\quad (22)$$

so that the boundary conditions on B are automatically satisfied.

It will be useful to employ the scalar and vector potentials ϕ , \underline{A} which satisfy the equations

$$\underline{E} = -\text{grad } \phi - \frac{\partial \underline{A}}{\partial t} \quad \underline{B} = \text{curl } \underline{A} \quad (23)$$

Now it is well known that the scalar potential of a stationary line charge $-(e/h)$ is given by

$$\phi = \frac{+e}{h\epsilon_0} \log_e |\underline{r} - \underline{r}'| \quad (24)$$

where \underline{r}' is the radius vector to the line charge, and \underline{r} is the radius vector to the point at which the field is being measured.

If the velocity of the line charge is very small compared with the velocity of light, we can ignore the difference between local and retarded time, and write the scalar and vector potentials of a line charge moving in free space

$$\begin{aligned}\phi &= \frac{+e}{\epsilon_0 h} \log_e |\underline{r} - \underline{r}'| \\ \underline{A} &= \frac{+e\mu_0}{h} \underline{v}' \log_e |\underline{r} - \underline{r}'|\end{aligned}\quad (25)$$

where $\underline{v}' = (\dot{r}', r'\dot{\theta}', 0)$ is the velocity of the line charge. These potentials are the Green's functions for the problem. Our task is now to find solutions of the charge-free Maxwell's equations, of the form of Eq. 22, regular everywhere within the anode-cathode interaction space, and in the slots, which, when added to the fields derived from Eq. 25, satisfy the boundary conditions stated above. We proceed as follows. The magnetron interior is divided into two regions, the anode-cathode interaction space and the slots. A general solution of Maxwell's equations for charge-free space is obtained in these regions as an infinite series of the appropriate orthogonal functions. The fields derived from the potentials of Eq. 25 are likewise expanded as a series of the orthogonal functions appropriate to the cylindrical interaction space. The boundary conditions, and the conditions for continuity, are sufficient to determine the unknown coefficients in these expansions.

Wherever it may be necessary, to avoid confusion, we shall denote the fields in the S th slot by the superscript S , the fields in the anode-cathode interaction space by the superscript A , and the fields derived from the Green's functions of Eqs. 25 by the

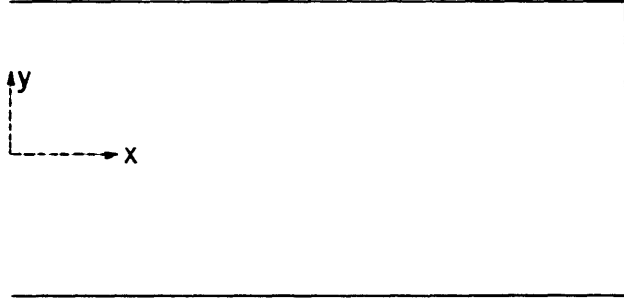


Fig. 2 Coordinate axes in rectangular anode slot.

superscript G.

We shall eliminate the time explicitly from Maxwell's equations by taking their Fourier transforms. In the remainder of this section, unless otherwise stated, all field quantities will be replaced by their Fourier transforms; formally this can be done by writing $\underline{E} e^{i\omega t}$, $\underline{B} e^{i\omega t}$, for \underline{E} and \underline{B} and dividing through by $e^{i\omega t}$.

1. The charge-free fields in the slots. The anode slots are assumed to be rectangular in shape, of width d and length L . It is best to express the fields in Cartesian coordinates; the y -axis lies in the open face of the slot, and the x -axis along the center of the slot as shown in Fig. 2. If we ignore all the evanescent modes it can be shown (2) that the fields in the slot may be written

$$\begin{aligned} H_z^S &= H_0^S \left[\exp\left(-\frac{i\omega x}{c}\right) + a \exp\left(\frac{i\omega x}{c}\right) \right] \\ E_x^S &= 0 \\ E_y^S &= \sqrt{\frac{\mu_0}{\epsilon_0}} H_0^S \left[\exp\left(-\frac{i\omega x}{c}\right) - a \exp\left(\frac{i\omega x}{c}\right) \right] \end{aligned} \quad (26)$$

The a in this equation will be determined by the boundary conditions at the closed end of the slot, and is independent of s because we have assumed that the slots are symmetrically loaded, as discussed in paragraph 3, page 14. The H_0^S are coefficients determined by the boundary conditions at the open end of the slot.

2. The charge-free fields in the anode-cathode interaction space. It can be shown that the charge-free fields in the anode-cathode interaction space are given by (3)

$$\begin{aligned} H_z^A &= \sum_{m=0}^{\infty} \left[A_m^A Z_{m1}\left(\frac{\omega r}{c}\right) \cos m\theta + B_m^A Z_{m2}\left(\frac{\omega r}{c}\right) \sin m\theta \right] \\ E_r^A &= \frac{m}{i\omega\epsilon_0} \sum_{m=0}^{\infty} \left[-A_m^A Z_{m1}\left(\frac{\omega r}{c}\right) \frac{\sin m\theta}{r} + B_m^A Z_{m2}\left(\frac{\omega r}{c}\right) \frac{\cos m\theta}{r} \right] \\ E_\theta^A &= +i\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{m=0}^{\infty} \left[A_m^A Z'_{m1}\left(\frac{\omega r}{c}\right) \cos m\theta + B_m^A Z'_{m2}\left(\frac{\omega r}{c}\right) \sin m\theta \right] \end{aligned} \quad (27)$$

where

$$\begin{aligned} Z_{m1}\left(\frac{\omega r}{c}\right) &= J_m\left(\frac{\omega r}{c}\right) + \gamma_{m1} N_m\left(\frac{\omega r}{c}\right) \\ Z_{m2}\left(\frac{\omega r}{c}\right) &= J_m\left(\frac{\omega r}{c}\right) + \gamma_{m2} N_m\left(\frac{\omega r}{c}\right) \end{aligned} \quad (28)$$

$J_m(x)$ and $N_m(x)$ are the Bessel and Neumann functions of order m respectively; γ_{m1} and γ_{m2} are real constants determined by the boundary conditions at the cathode; A_m^A and B_m^A are real constants determined by the boundary conditions at the anode.

3. The fields derived from the potential Green functions. Equation 25 may be written

$$\begin{aligned}\phi &= \frac{e}{\epsilon_0 h} \log_e \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \\ \underline{A} &= \frac{e\mu_0}{h} \underline{r}' \log_e \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \end{aligned} \quad (29)$$

and the corresponding fields may be derived from Eq. 23. Since we assume that the velocity of the line charge is very small compared with the velocity of light, we may put

$$\underline{E}^G = -\text{grad } \phi^G.$$

In Appendix I of Section III we have obtained the Fourier transforms of these field quantities and get

$$\begin{aligned}H_z^G(i\omega) &= -\frac{ei\omega}{h} \sum_{m=1}^{\infty} \left[\frac{F_{mc}(i\omega)}{r^m} \frac{\sin m\theta}{m} - \frac{F_{ms}(i\omega)}{r^m} \frac{\cos m\theta}{m} \right] \quad r > r' \\ &= +\frac{ei\omega}{h} \sum_{m=1}^{\infty} \left[r^m G_{mc}(i\omega) \frac{\sin m\theta}{m} - r^m G_{ms}(i\omega) \frac{\cos m\theta}{m} \right] \quad r < r' \end{aligned} \quad (30)$$

$$\begin{aligned}E_z^G(i\omega) &= \frac{-e}{\epsilon_0 rh} \sum_{m=1}^{\infty} \left[\frac{F_{mc}(i\omega)}{r^m} \sin m\theta - \frac{F_{ms}(i\omega)}{r^m} \cos m\theta \right] \quad r > r' \\ &= \frac{-e}{\epsilon_0 rh} \sum_{m=1}^{\infty} \left[G_{mc}(i\omega) r^m \sin m\theta - r^m G_{ms}(i\omega) \cos m\theta \right] \quad r < r' \end{aligned} \quad (31)$$

where

$$\begin{aligned}F_{mc}(i\omega) &= \int_{-\infty}^{\infty} (r')^m \cos m\theta' \exp(-i\omega t) dt; \quad G_{mc}(i\omega) = \int_{-\infty}^{\infty} \left(\frac{1}{r'}\right)^m \cos m\theta' \exp(-i\omega t) dt \\ F_{ms}(i\omega) &= \int_{-\infty}^{\infty} (r')^m \sin m\theta' \exp(-i\omega t) dt; \quad G_{ms}(i\omega) = \int_{-\infty}^{\infty} \left(\frac{1}{r'}\right)^m \sin m\theta' \exp(-i\omega t) dt. \end{aligned} \quad (32)$$

With the aid of Eqs. 26, 27, 30, and 31 we can use the boundary conditions to determine the various unknowns: From the requirement that the tangential electric field be zero over the cathode surface, we obtain the equation*

$$E_{\theta}^G(i\omega) + E_{\theta}^A(i\omega) = 0 \quad \text{at } r = r_c. \quad (33)$$

*For a discussion of these boundary conditions see page 14. For a definition of the interpretation of the superscripts on the field quantities see page 15.

From the requirement that the tangential electric field be zero over the portions of the anode that lie between the slots, we obtain the equation

$$E_{\theta}^G(i\omega) + E_{\theta}^A(i\omega) = 0 \quad (34)$$

for $r = r_a$ and for values of θ satisfying the inequality

$$\frac{2s\pi}{N} + \frac{d}{2r_a} < \theta < \frac{(2s+1)\pi}{N} - \frac{d}{2r_a} \quad (s = 0, 1, \dots, N-1).$$

From the requirement that the tangential electric field be continuous across the mouth of the slots, we obtain the equation

$$E_{\theta}^G(i\omega) + E_{\theta}^A(i\omega) = E_y^S(i\omega) \quad (35)$$

for $r = r_a$, and for values of θ satisfying the inequality

$$\frac{2s\pi}{N} - \frac{d}{2r_a} < \theta < \frac{2s\pi}{N} + \frac{d}{2r_a} \quad (s = 0, 1, \dots, N-1).$$

Finally from the requirement that the magnetic field should be continuous across the center of the slot we obtain

$$H_z^G(i\omega) + H_z^A(i\omega) = H_z^S(i\omega) \quad (36)$$

for $r = r_a$ and for $\theta = 2s\pi/N$ ($s = 0, 1, \dots, N-1$). In these equations r_c is the cathode radius, r_a is the anode radius, and d is the width of the slot, as shown in Fig. 1.

The various field quantities in Eqs. 33 to 36 are given by Eqs. 26, 27, 30 and 31 and between them it is possible to express all the unknown field quantities in terms of quantities that depend only upon the orbit of the electron. In the present case we are concerned with the fields in the slots which are in turn determined by the quantity H_o^S . If the necessary algebraic elimination be carried through it can be shown that H_o^S is given by the rather formidable expression

$$H_o^S(1 + a) = \sum_{m=0}^{\infty} \frac{ei}{\sqrt{\mu_o \epsilon_o}} \frac{1}{h} \frac{\Delta_{m1}}{\Delta_{m2}} \quad (37)$$

where

$$\Delta_{m2} = \left[J'_m\left(\frac{\omega r_a}{c}\right) - \frac{N'_m\left(\frac{\omega r_a}{c}\right) J'_m\left(\frac{\omega r_c}{c}\right)}{N'_m\left(\frac{\omega r_c}{c}\right)} \right] - \frac{iN(1-a)}{2m\pi(1+a)} \sin\left(\frac{md}{2r_a}\right) \left[J_m\left(\frac{\omega r_a}{c}\right) - \frac{N_m\left(\frac{\omega r_a}{c}\right) J'_m\left(\frac{\omega r_c}{c}\right)}{N'_m\left(\frac{\omega r_c}{c}\right)} \right]$$

and

$$\Delta_{ml} = \left\{ \cos \left(\frac{2ms\pi}{N} \right) \left[\frac{F_{ms}(i\omega)}{r_a^{m+1}} - \frac{N'_m \left(\frac{\omega r_a}{c} \right)}{N'_m \left(\frac{\omega r_c}{c} \right)} r_c^{m-1} G_{ms}(i\omega) \right] \right. \\ \left. - \sin \left(\frac{2ms\pi}{N} \right) \left[\frac{F_{mc}(i\omega)}{r_a^{m+1}} - \frac{N'_m \left(\frac{\omega r_a}{c} \right)}{N'_m \left(\frac{\omega r_c}{c} \right)} r_c^{m-1} G_{mc}(i\omega) \right] \right\} \\ \times \left\{ \frac{\omega r_a}{mc} \left[J'_m \left(\frac{\omega r_a}{c} \right) - \frac{N'_m \left(\frac{\omega r_a}{c} \right)}{N'_m \left(\frac{\omega r_c}{c} \right)} J'_m \left(\frac{\omega r_c}{c} \right) \right] + \left[J_m \left(\frac{\omega r_a}{c} \right) - \frac{N_m \left(\frac{\omega r_a}{c} \right) J'_m \left(\frac{\omega r_c}{c} \right)}{N'_m \left(\frac{\omega r_c}{c} \right)} \right] \right\}.$$

This complicated result can be much simplified when the inequality

$$\frac{\omega r_a}{c} = \frac{2\pi r_a}{\lambda} \ll 1$$

is satisfied, which is certainly the case with the magnetron of Fig. 1 where

$$\lambda \approx 10^{-1}, \quad r_a = 2.85 \times 10^{-3} \quad \text{and} \quad \frac{2\pi r_a}{\lambda} \approx 0.1.$$

In this case we can use the Laurent series expansions for $J_m(x)$ and $N_m(x)$ and neglect all terms except the first; so that

$$J_m(x) \approx \frac{x^m}{m! 2^m}, \quad N_m(x) \approx -\frac{(m-1)!}{\pi} \left(\frac{2}{x} \right)^m \quad (38)$$

are valid for $m > 1$ and $x \ll 1$. We now have, on substituting in Eq. 37

$$H_o^S = \sum_{m=0}^{\infty} \frac{ei}{h(1+a)\sqrt{\epsilon_o \mu_o}} \left\{ \frac{2 \left(\frac{\omega r_a}{c} \right) \left[1 - \left(\frac{r_c}{r_a} \right)^{2m} \right] \left[\frac{X_{ms}}{r_a^{m+1}} \cos \left(\frac{2\pi ms}{N} \right) - \frac{X_{mc}}{r_a^{m+1}} \sin \left(\frac{2\pi ms}{N} \right) \right]}{m \left[1 - \left(\frac{r_c}{r_a} \right)^{2m} \right] - \frac{iN}{2\pi m} \left(\frac{1-a}{1+a} \right) \frac{\omega r_a}{c} \left[1 + \left(\frac{r_c}{r_a} \right)^{2m} \right] \sin \left(\frac{md}{2r_a} \right)} \right\} \quad (39)$$

where

$$X_{mc}(i\omega) = \int_{-\infty}^{\infty} (r')^m \left[1 - \left(\frac{r_c}{r'} \right)^{2m} \right] \cos m\theta' \exp(-i\omega t) dt \\ X_{ms}(i\omega) = \int_{-\infty}^{\infty} (r')^m \left[1 - \left(\frac{r_c}{r'} \right)^{2m} \right] \sin m\theta' \exp(-i\omega t) dt. \quad (40)$$

In the magnetron of Fig. 1 $(r_c/r_a) = 1/1.6$ and, as the dominant mode is that for which $m=8$, it will be possible to neglect $(r_c/r_a)^{2m}$ in comparison with unity. The sole effect of the cathode therefore is to provide a screening factor of the form $1 - (r_c/r)^{2m}$ which

is effectively unity save very near the cathode.

For $m \leq 8$, $\sin(md/2r_a) \approx md/2r_a$. Accordingly* for $m \leq 8$ the expression for H_o^S assumes the simplified approximate form

$$H_o^S = \sum_{m=0}^{\infty} \frac{ei}{h(1+a)\sqrt{\epsilon_o \mu_o}} \left\{ \frac{\left(\frac{2}{m}\right)\left(\frac{\omega r_a}{c}\right) \left[\frac{X_{ms}}{r_a^{m+1}} \cos \frac{2ms\pi}{N} - \frac{X_{mc}}{r_a^{m+1}} \sin \frac{2ms\pi}{N} \right]}{1 - \frac{iN}{4m\pi} \frac{1-a}{1+a} \frac{\omega d}{c}} \right\} \quad (41)$$

where X_{mc} and X_{ms} are given by Eq. 40.

Using Eq. 41 and Eq. 26 we see that we have succeeded in determining the electromagnetic fields set up in the slot by an arbitrarily moving line charge. We have thus completed the first part of our task and are now in a position to apply Campbell's theorem to determine the mean square fluctuation produced across the output load.

D. Campbell's Theorem for the Temperature Limited Magnetron. As stated in Section III the effect produced by an electron is virtually independent of the initial thermal velocity; only the initial angular coordinate is important. Accordingly we must apply Campbell's theorem in the form applicable to the case where there is a single continuum of fundamental events as in Eq. 10.

Let $\lambda d\theta dt/2\pi$ be the probability that an electron be emitted in the time interval dt in the angular range $d\theta$ so that, because λ is assumed to be independent of θ , $\lambda = AI_c/e$ where A is the total cathode area, and I_c is the emitted current density in amps/meter².

Let $V_\theta(t)$ be the voltage produced across the output load by an electron emitted at time $t=0$ with initial angular coordinate in the range $\theta_o < \theta < \theta_o + d\theta$, and let $V_\theta(i\omega)$ be the Fourier transform of $V_\theta(t)$ delivered as in Eq. 6. Then Campbell's theorem, as stated in Eq. 10, gives us

$$\begin{aligned} \overline{[V(t) - \overline{V(t)}]^2} &= 2\lambda \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^\infty |V_\theta(i\omega)|^2 d\nu \\ &= \frac{2AI_c}{e} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^\infty |V_\theta(i\omega)|^2 d\nu \end{aligned} \quad (42)$$

We deduce that the average noise power developed, which may be defined as

$$\frac{\overline{[V(t) - \overline{V(t)}]^2}}{R}$$

*This is not always a valid result. It is valid with the particular numerical values of the experimental tube discussed in this paper (see Fig. 1). Since this tube has 16 slots, $m = 8$ corresponds to the π -mode.

is given by

$$\frac{\overline{[V(t) - \overline{V(t)}]^2}}{R} = \frac{2AI_c}{e} \int_0^{2\pi} \frac{d\theta_o}{2\pi} \int_0^\infty \frac{|V_\theta(i\omega)|^2}{R} d\nu \quad (43)$$

where R is the parallel resistance component at the output load. The power $P^S(\nu)$ dissipated in the load in the S th slot in unit bandwidth is given, accordingly, by

$$P^S(\nu) d\nu = \frac{2AI_c}{e} \int_0^{2\pi} \frac{|V_\theta(i\omega)|^2}{R} \frac{d\theta_o}{2\pi} d\nu \quad (44)$$

Because of the symmetry assumption, the noise power dissipated in any one load is equal on the average to the noise power dissipated in any other load. Hence $P^S(\nu)$ is independent of S , and $P(\nu)$ the total power dissipated in the loads, in unit bandwidth, is given by

$$P(\nu) = N P^S(\nu) = \frac{2NAI_c}{e} \int_0^{2\pi} \frac{|V_\theta(i\omega)|^2}{R} \frac{d\theta_o}{2\pi} \quad (45)$$

where N is the number of the slots.

$P(\nu)$ is the quantity that we are primarily concerned to find. As a first step let us consider the quantity

$$p_\theta^S(\omega) = \frac{1}{2R} |V_\theta(i\omega)|^2 = \frac{1}{2R} V_\theta(i\omega) V_\theta^*(i\omega) \quad (46)$$

This can be interpreted as the power dissipated, at angular frequency ω , in the load resistance, by an electron emitted from points $(r_c \theta_o)$ on the cathode. Now we have assumed that no power is dissipated in the walls of the slot so that this power must also equal the power flowing across the open face of the slot. This power is equal to (1)

$$\text{Real part of } \frac{1}{2} \int E^S(i\omega) \times \underline{\underline{H}}^S(i\omega) dS$$

where the integration is taken over the open face of the slot, $E^S(i\omega)$ is the Fourier transform of the electric vector in the slot, and $\underline{\underline{H}}^S(i\omega)$ is the conjugate complex of the transform of the magnetic vector in the slot. $\underline{\underline{E}}^S(i\omega)$ and $\underline{\underline{H}}^S(i\omega)$ are given by Eq. 26 with $x=0$. Hence we have

$$\begin{aligned} p_\theta^S(\omega) &= \text{Re} \left\{ \frac{1}{2} \int_{\text{over slot face at } x=0} \underline{\underline{E}} \times \underline{\underline{H}}^* dS \right\} \\ &= \text{Re} \left\{ \frac{1}{2} \sqrt{\frac{\mu_o}{\epsilon_o}} \int_{-d/2}^{d/2} dy \int_0^h dz |H_o^S|^2 (1 - \alpha)(1 + \alpha^*) \right\} \\ &= \frac{hd}{2} \sqrt{\frac{\mu_o}{\epsilon_o}} |H_o^S|^2 \text{Re} \left\{ (1 - \alpha)(1 + \alpha^*) \right\} \\ &= \frac{hd}{2} \sqrt{\frac{\mu_o}{\epsilon_o}} (1 - \alpha\alpha^*) |H_o^S|^2 \end{aligned} \quad (47)$$

where H_0^S is given by Eq. 41 for $m \approx 8$, and more generally by Eq. 39 for $m < 8$.

From Eq. 43 we can see that it is $p^S(\omega)$, the average value of $p_\theta^S(\omega)$, with which we are primarily concerned, where

$$p^S(\omega) = \int_0^{2\pi} p_\theta^S(\omega) \frac{d\theta}{2\pi} = \frac{hd}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} (1 - \alpha\alpha^*) \int_0^{2\pi} \frac{|H_0^S|^2}{2\pi} d\theta_0 \quad (48)$$

and α is independent of θ .

H_0^S is given as an infinite series summed over the complete set of modes that can be set up in the anode-cathode interaction space. Hence $p^S(\omega)$ must be expressed as a doubly infinite sum containing all the cross-product terms. However $p^S(\omega)$ will contain only the squared terms, as all the cross-product terms will average to zero and will be of the form

$$p^S(\omega) = \frac{hd}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} (1 - \alpha\alpha^*) \sum_{m=1}^{\infty} |H_{0m}|^2. \quad (49)$$

This important result, which is proved in Appendix II of Section III, states that the output noise power of the magnetron may be regarded as a sum of the noise powers associated with the individual modes of the magnetron. More specifically we have the following lemma.

Lemma 1. If the emissivity of the cathode of a cylindrical magnetron be uniform, the noise power associated with any one mode is statistically independent of the noise power in any other mode. The total output noise power is equal to the sum of the noise powers associated with the individual modes. It should be emphasized that this result is true only when the emissivity is uniform. Where it is not uniform, the modes become mixed up, and some or all of the cross-product terms are non-zero. It is very doubtful whether a small asymmetry in the cathode emission would have any appreciable effect on the output noise power.

From Appendix II we have that $P(\nu)$, the total noise power dissipated in the N output loads, is given by

$$\begin{aligned} P(\nu) &= N P^S(\nu) = \frac{4NAI_c}{e} p^S(\nu) \\ &= 2e I_c A \frac{Nd}{2h} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1 - \alpha\alpha^*}{(1 + \alpha)(1 + \alpha^*)} \sum_{m=1}^{\infty} \left\{ \left(\frac{2}{m} \right)^2 \omega^2 \left(\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \right) \right\} \\ &\quad \left\{ \left| \frac{1}{1 - \frac{iN}{4m\pi} \frac{\omega d}{c} \left(\frac{1 - \alpha}{1 + \alpha} \right)} \right|^2 \right\} \end{aligned} \quad (50)$$

(which is independent of h since $A = 2\pi r_c \cdot h$ where r_c is the cathode radius) where Y_{mc} and Y_{ms} are defined by the equations

$$\begin{aligned}
Y_{mc} &= \int_{-\infty}^{\infty} (r)^m \left[1 - \frac{r_c}{r} \right]^{2m} \cos m\theta \exp(-i\omega t) dt \\
Y_{ms} &= \int_{-\infty}^{\infty} (r)^m \left[1 - \frac{r_c}{r} \right]^{2m} \sin m\theta \exp(-i\omega t) dt
\end{aligned} \tag{51}$$

and where (r, θ) are the coordinates, at time t of an electron emitted, at time $t=0$, from the point $(r_c, 0)$ on the cathode.

On the face of it this is a surprising result. It implies that increasing the length of the cathode and, consequently, increasing the emitted cathode current will not affect the noise power, which is certainly not what one would expect by analogy with a conventional tube. The result comes about, of course, because the field produced by the electron in the slot is inversely proportional to the length of the magnetron axially. The power flowing into the slot from one electron is inversely proportional to the magnetron length since it is proportional to the product of field² \times area of slot face. The total noise power, since it is the product of the power produced by one electron and the total emission current, is independent of the magnetron length.

Inspection shows that Y_{mc} and Y_{ms} are functions only of the orbit followed by the electron. To complete our analysis it is necessary only to find a which, as we stated above, following Eq. 26, is determined by the boundary conditions at the closed end of the slot. We have assumed (paragraph 3, page 14) that the termination of the closed end of the slot is purely resistive, equal to R_t for example. If impedance be defined as the ratio (E_y/H_z) , we have

$$R_t = \left(\frac{E_y^s}{H_z^s} \right)_{x=L}$$

where L is the length of the slot. From Eq. 26 this gives us

$$R_t = \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{\exp\left(\frac{-i\omega L}{c}\right) - a \exp\left(\frac{i\omega L}{c}\right)}{\exp\left(\frac{-i\omega L}{c}\right) + a \exp\left(\frac{i\omega L}{c}\right)}$$

where $\sqrt{\mu_o/\epsilon_o} = Z_o = 377$ is the characteristic impedance of free space. Solving for a we get

$$a = \frac{Z_o - R_t}{Z_o + R_t} \exp\left(\frac{-2i\omega L}{c}\right) .$$

But R_t is not itself a measurable quantity. The quantity that we should prefer to use is Q_m , the Q of the magnetron measured at the first resonance associated with the m^{th} mode. As is well known, the output admittance of a magnetron in the neighborhood of resonance is very closely that of a parallel damped tuned circuit, the inverse fractional half power bandwidth of which is equal to Q_m . It is shown in Appendix III that

$$Q_m \approx \frac{Z_o}{R_t} \frac{\pi}{4} \quad (52)$$

and that for frequencies in the neighborhood of ν_{om} , the first resonance associated with the m^{th} mode, the total noise power delivered to the loads is of the form

$$P(\nu) d\nu = \sum_{m=1}^{\infty} 2e I_c A \frac{Nd}{2h} \frac{Z_o^4 Q_m}{\pi} \left(\frac{Z}{m}\right)^2 \omega^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \frac{d\nu}{\left[1 + Q_m^2 \left(\frac{\nu}{\nu_{om}} - \frac{\nu_{om}}{\nu}\right)^2\right]} \quad (53)$$

In practice, if we are measuring the noise power in the neighborhood of the resonant frequency ν_{om} , only this term in the infinite series of Eq. 42 will be important and we will have

$$P(\nu) d\nu = 2e A I_c \frac{Nd}{2h} Z_o \frac{4 Q_m}{\pi} \left(\frac{Z}{m}\right)^2 \omega^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \frac{d\nu}{\left[1 + Q_m^2 \left(\frac{\nu}{\nu_{om}} - \frac{\nu_{om}}{\nu}\right)^2\right]} \quad (54)$$

while

$$P(\nu_{om}) = 2e A I_c \frac{Nd}{2h} Z_o \frac{4 Q_m}{\pi} \left(\frac{Z}{m}\right)^2 \omega^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \quad (55)$$

In arriving at this result it was assumed that all the noise power was dissipated in the external load, and that there were no losses to the walls. This will not in general be the case; the total Q of the output circuit is composed partly of Q_L , the external Q , and partly of Q_{om} , the unloaded magnetron Q , where

$$Q_m = \frac{Q_L Q_{om}}{Q_{om} + Q_L}$$

and the fraction of the total power dissipated in the load itself will be

$$\frac{Q_{om}}{Q_{om} + Q_L}.$$

Hence the total noise power dissipated in the load resistance at resonance in a practical case in bandwidth $d\nu$ is

$$P(\nu_{om}) = 2e I_c A \frac{Nd}{2h} Z_o \frac{4}{\pi} \frac{Q_m^2}{Q_L} \left(\frac{Z}{m}\right)^2 \omega_{om}^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}}$$

where Q is the total and Q_L the external Q .

A measure of the "noisiness" of the magnetron can be obtained by comparing $P(\nu_{om})d\nu$, the total noise power delivered to the load, with $kT_r d\nu$, the noise power generated in the load by thermal agitation. The ratio ρ of these two quantities is given by

$$\rho = \frac{P(\nu_{om})}{kT_r} = \frac{2eI_c A}{kT_r} \frac{4}{\pi} Z_o \frac{Q_m^2}{Q_L} \frac{Nd}{2h} \left(\frac{2}{m}\right)^2 \omega_{om}^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \quad (56)$$

where e is the electronic charge

k is Boltzmann's constant

T_r is the room temperature

Z_o is 377Ω the impedance of free space

Q_m is the total magnetron Q

Q_L is the external Q

N is the number of slots

d is the width of a slot

m is the number of the mode

r_a is the magnetron anode radius

ω is the angular resonant frequency

h is the magnetron length.

E. Equivalent Noise Circuit. In this section we shall set up an equivalent noise circuit for the temperature limited magnetron. It should be emphasized that this can often be done in a number of ways, and that the value of such a circuit is purely empirical; it does not necessarily have any significance.

If impedance is defined as the ratio E_y/H_z , the input impedance to the s th slot can be written

$$Z_I^S = Z_o \left(\frac{1-a}{1+a} \right) = R_I + iX_I \quad (57)$$

where R and X are real, from Eq. 26 with x equal to zero. Taking real and imaginary parts of Z_I^S we have

$$R_I = Z_o \frac{1-aa^*}{(1+a)(1+a^*)} \quad (58)$$

$$X_I = Z_o \frac{a^* - a}{(1+a)(1+a^*)} \quad (59)$$

The form of these expressions and the fact that in the symmetrically loaded magnetron the N slots are effectively in parallel suggests that we look for an equivalent circuit of the form shown in Fig. 3 where the noise current generator is in parallel with an admittance iY_s , representing the admittance of the cathode-anode interaction space, and N impedances each equal to the input impedance of a slot.

If the circuit is to be of any real value it must be possible to find $i_s(\nu)$ and $iY_s(\nu)$ independent of a (i.e. independent of the magnetron load) such that the total power delivered to the N loads is as given by Eq. 50. We shall show that this is possible if we make one additional restriction.

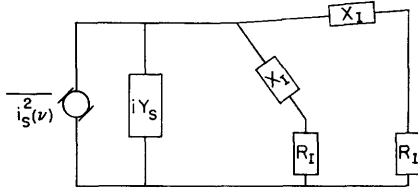


Fig. 3

Equivalent circuit for noise in a single mode of a temperature limited magnetron (only two of the N slot circuits are marked).

By inspection of Fig. 3 we see that the total power delivered to the loads is

$$P'(\nu) = \frac{N \overline{i_s^2(\nu)} R_I}{N^2 \left| 1 + \frac{i Y_s (R_I + i X_I)}{N} \right|^2} d\nu \quad (60)$$

Using Eqs. 57 and 58 we see that Eq. 60 can be put in the form

$$P'(\nu) = \frac{\overline{i_s^2(\nu)}}{N} \frac{1 - \alpha \alpha^*}{(1 + \alpha)(1 + \alpha^*)} \frac{Z_o}{\left| 1 + \frac{i Y_s Z_o}{N} \frac{1 - \alpha}{1 + \alpha} \right|^2} d\nu \quad (61)$$

Reference to Eq. 50 shows that $P'(\nu)$ is identical with the output noise power associated with the m th mode for all α if we take

$$Y_s = \frac{-N^2}{4m\pi} \frac{\omega d}{c} \frac{1}{Z_o} \quad (62)$$

$$\overline{i_s^2(\nu)} = 2e A I_c \frac{N^2 d}{2h} \left(\frac{2}{m}\right)^2 \omega^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \quad (63)$$

The nature of the restriction mentioned above now becomes clear. Y_s is itself a function of m so that our equivalent circuit cannot be adopted to yield a representation for the total output noise of the magnetron; a separate circuit is needed for the noise associated with each mode. This is not a very serious limitation. In most cases we are interested only in the noise power associated with a given mode, and in the general case all that we have to do is to calculate the output noise power associated with each mode separately, and sum over all the modes to find the total output noise power. With the aid of the steady state theory for the magnetron given in Technical Report No. 118 we can obtain numerical estimates for the shot noise power delivered to the load of a magnetron, which we can compare with experiment.

APPENDIX I.

The Fourier Transforms of the Electromagnetic Field Components Associated with the Green's Function Potentials. The fields associated with the Green's potentials of Eq. 29 are

$$\begin{aligned} \underline{\underline{E}} &= -\text{grad } \phi = \left[\frac{-e}{\epsilon_o} \frac{r - r' \cos(\theta - \theta')}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}, \frac{-e}{\epsilon_o} \frac{r' \sin(\theta - \theta')}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}, 0 \right] \\ \underline{\underline{B}} &= \text{curl } \underline{\underline{A}} = \left[0, 0, -\mu_o e \frac{\dot{r}' r \sin(\theta - \theta') + r \dot{\theta}' [r' - r \cos(\theta - \theta')]}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \right] \end{aligned}$$

Let us utilize the identity

$$\begin{aligned} \left[r^2 + r'^2 - 2rr' \cos(\theta - \theta') \right]^{-1} &= (r^2 - r'^2)^{-1} \left[1 + 2 \sum_{m=1}^{\infty} \left(\frac{r'}{r} \right)^m \cos m(\theta - \theta') \right] \quad r > r' \\ &= (r^2 - r'^2)^{-1} \left[1 + 2 \sum_{m=1}^{\infty} \left(\frac{r}{r'} \right)^m \cos m(\theta - \theta') \right] \quad r' > r \end{aligned}$$

and for the moment, confine attention to the region $r > r'$. The fields in the region $r' > r$ can be written by interchanging r and r' , and we have

$$\begin{aligned} E_{\theta}^G &= \frac{-er' \sin(\theta - \theta')}{\epsilon_0(r^2 - r'^2)} \left\{ 1 + \sum_{m=1}^{\infty} 2 \left(\frac{r'}{r} \right)^m \cos m(\theta - \theta') \right\} \\ &= \frac{-e}{\epsilon_0 r} \sum_{m=0}^{\infty} \left[\left(\frac{r'}{r} \right)^m \cos m\theta' \sin m\theta - \left(\frac{r'}{r} \right)^m \sin m\theta' \cos m\theta \right] \end{aligned}$$

$$\begin{aligned} H_z^G(t) &= \frac{\epsilon_0 r \dot{r}'}{r'} E_{\theta}^G(t) + e \dot{\theta}' \sum_{m=0}^{\infty} \left(\frac{r'}{r} \right)^m \cos m(\theta - \theta') \\ &= -e \sum_{m=1}^{\infty} \frac{\sin m\theta}{r^m} \left[(r')^{m-1} \dot{r}' \cos m\theta' - (r')^m \dot{\theta}' \sin m\theta' \right] \\ &\quad - \frac{\cos m\theta}{r^m} \left[(r')^{m-1} \dot{r}' \sin m\theta' + (r')^m \dot{\theta}' \cos m\theta' \right] \\ &= -e \sum_{m=1}^{\infty} \left\{ \frac{\sin m\theta}{r^m} \frac{d}{dt} \left[\frac{(r')^m \cos m\theta'}{m} \right] - \frac{\cos m\theta}{r^m} \frac{d}{dt} \left[\frac{(r')^m \sin m\theta'}{m} \right] \right\}. \end{aligned}$$

We must now take the Fourier transforms of these fields where $F(i\omega)$, the Fourier transform of $f(t)$, is defined by the reciprocal set of relations

$$\begin{aligned} F(i\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t} d\omega \end{aligned}$$

If

$$F_{mc}(i\omega) = \int_{-\infty}^{\infty} (r')^m \cos m\theta' \exp -i\omega t dt$$

$$F_{ms}(i\omega) = \int_{-\infty}^{\infty} (r')^m \sin m\theta' \exp(-i\omega t) dt$$

and

$$E_{\theta}^G(i\omega) = \int_{-\infty}^{\infty} E_{\theta}^G(t) e^{-i\omega t} dt$$

$$H_Z^G(i\omega) = \int_{-\infty}^{\infty} H_Z^G(t) e^{-i\omega t} dt$$

then

$$E_{\theta}^G(i\omega) = \frac{-e}{\epsilon_0 r} \sum_{m=0}^{\infty} \left[\frac{F_{mc}(i\omega) \sin m\theta}{r^m} - \frac{F_{ms}(i\omega) \cos m\theta}{r^m} \right] \quad r > r' .$$

Remembering that the Fourier transform of df/dt , is $i\omega$ times the Fourier transform of $f(t)$, we have

$$H_Z^G(i\omega) = -\frac{ei\omega}{m} \sum_{m=1}^{\infty} \left[\frac{F_{mc}(i\omega) \sin m\theta}{r^m} - \frac{F_{ms}(i\omega) \cos m\theta}{r^m} \right] \quad r > r' .$$

In the region $r < r'$ we have

$$E_{\theta}^G(i\omega) = \frac{-e}{\epsilon_0 r} \sum_{m=0}^{\infty} \left[r^m G_{mc}(i\omega) \sin m\theta - r^m G_{ms}(i\omega) \cos m\theta \right] \quad r < r'$$

$$H_Z^G(i\omega) = \frac{ei\omega}{m} \sum_{m=1}^{\infty} \left[r^m G_{mc}(i\omega) \sin m\theta - r^m G_{ms}(i\omega) \cos m\theta \right] \quad r < r'$$

where

$$G_{mc}(i\omega) = \int_{-\infty}^{\infty} (r')^{-m} \cos m\theta' \exp -i\omega t dt$$

$$G_{ms}(i\omega) = \int_{-\infty}^{\infty} (r')^{-m} \sin m\theta' \exp -i\omega t dt .$$

These results are quoted in Eqs. 30, 31 and 32.

APPENDIX II

In this appendix we shall show that $p^S(\omega)$, as defined by Eq.48, is of the general form of Eq. 49 where H_0^S is given by Eq.39.

In order to simplify the arithmetic let us confine attention to the power delivered to the slot for which $s = 0$. This choice obviously involves no loss of generality. By inspection of Eq. 39 we see that

$$H_o^0 = H_o = \sum_{m=1}^{\infty} a_m X_m$$

where the a_m are coefficients that do not depend on θ_o , the initial angular coordinates of the electron.

If (r, θ) be the coordinates, at time t , of an electron emitted at time $t = 0$, at (r_c, o) , and if (r', θ') be the coordinates of an electron emitted at time $t = 0$ at (r_c, θ_o) , then

$$r' = r, \quad \theta' = \theta + \theta_o$$

so that

$$\begin{aligned} X_{ms} &= \int_{-\infty}^{\infty} (r')^m \left[1 - \left(\frac{r_c}{r} \right)^{2m} \right] \sin m\theta' \exp(-i\omega t) dt \\ &= \sin m\theta_o \int_{-\infty}^{\infty} (r)^m \left[1 - \left(\frac{r_c}{r} \right)^{2m} \right] \cos m\theta \exp(-i\omega t) dt \\ &\quad + \cos m\theta_o \int_{-\infty}^{\infty} (r)^m \left[1 - \left(\frac{r_c}{r} \right)^{2m} \right] \sin m\theta \exp(-i\omega t) dt . \end{aligned}$$

Now

$$|H_o|^2 = \sum_{m=1}^{\infty} a_m X_m \sum_{n=1}^{\infty} a_n^* X_n^*$$

so that

$$|H_o|^2 = \sum_{m=1}^{\infty} a_m a_m^* X_{ms} X_{ms}^* + 2 \sum_{n \neq m} \sum_{m=1}^{\infty} a_n a_m^* X_{ms}^* X_{ns} .$$

But

$$p^S(\omega) = \frac{\hbar d}{2} \sqrt{\frac{\mu_o}{\epsilon_o}} (1 - \alpha\alpha^*) \int_0^{2\pi} \frac{|H_o^S|^2}{2\pi} d\theta_o .$$

Hence if

$$\begin{aligned} |Y_{mc}| &= \left| \int_{-\infty}^{\infty} (r)^m \left[1 - \left(\frac{r_c}{r} \right)^{2m} \right] \cos m\theta \exp(-i\omega t) dt \right| \\ |Y_{ms}| &= \left| \int_{-\infty}^{\infty} (r)^m \left[1 - \left(\frac{r_c}{r} \right)^{2m} \right] \sin m\theta \exp(-i\omega t) dt \right| \end{aligned}$$

$$p^o(\omega) = \frac{dh}{2} \sqrt{\frac{\mu_o}{\epsilon_o}} (1 - aa^*) \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[a_m a_n^* (\sin m\theta_o Y_{mc} + \cos m\theta_o Y_{ms}) \right. \\ \left. (\sin n\theta_o Y_{nc}^* + \cos n\theta_o Y_{ns}^*) \right] .$$

Now

$$\frac{1}{2\pi} \int_0^{2\pi} \sin m\theta_o \cos n\theta_o d\theta_o = 0$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \cos m\theta_o \cos n\theta_o d\theta_o = \frac{1}{2\pi} \int_0^{2\pi} \sin m\theta_o \sin n\theta_o d\theta_o = \frac{1}{2} \delta_{mn} .$$

Hence

$$p^o(\omega) = \frac{hd}{2} \sqrt{\frac{\mu_o}{\epsilon_o}} (1 - aa^*) \frac{1}{2} \sum_{m=1}^{\infty} a_m a_m^* (|Y_{mc}|^2 + |Y_{ms}|^2)$$

which is of the form of Eq. 49.

Substituting for $p^o(\omega)$ in Eq. 44 and using Eq. 41 to yield an expression for a_m , we have

$$P(\nu) = N P^S(\nu) = \frac{4NAI_c}{e} p^S(\omega) = 2eI_c A \frac{Nd}{2h} \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{1 - aa^*}{(1+a)(1+a^*)} \\ \sum_{m=1}^{\infty} \left\{ \left(\frac{2}{m} \right)^2 \omega^2 \left(\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \right) \left| \frac{1}{1 - \frac{iN}{4m\pi} \frac{\omega d}{c} \left(\frac{1-a}{1+a} \right)} \right|^2 \right\} .$$

APPENDIX III

In this Appendix we will show that, in the neighborhood of the m^{th} mode resonance frequency ν_{om} , Eq. 50 is equivalent to Eq. 53. By comparison of these two equations we see that we must prove

$$\sqrt{\frac{\mu_o}{\epsilon_o}} \frac{1 - aa^*}{(1+a)(1+a^*)} \left\{ \left| \frac{1}{1 - \frac{iN}{4m\pi} \frac{\omega d}{c} \left(\frac{1-a}{1+a} \right)} \right|^2 \right\} = Z_o \frac{4Q_m}{\pi} \frac{1}{\left[1 + Q_m^2 \left(\frac{\nu}{\nu_{om}} - \frac{\nu_{om}}{\nu} \right)^2 \right]} \dots \quad (A-1)$$

for $\nu \approx \nu_{om} = (\omega_{om}/2\pi)$.

Following Eq. 57, we may write

$$Z_I^S = Z_o \left(\frac{1-a}{1+a} \right) = R_I + i X_I \quad (A-2)$$

where R_I and X_I are given by Eqs. 58 and 59. Accordingly we see that the left hand side of Eq. A-1 is equal to

$$R_I \left| \frac{1}{1 - \frac{iN}{4m\pi} \frac{\omega d}{c} \frac{Z_I}{Z_O}} \right|^2 \quad (A-3)$$

and will have a resonance when

$$1 + \frac{N}{4m\pi} \frac{\omega d}{c} \frac{X_I}{Z_O} = 0 \quad (A-4)$$

and be equal to

$$R_I \left| \frac{1}{\frac{R_I}{Z_O} - \frac{N(\frac{\omega d}{c})}{4m\pi}} \right|^2 \quad (A-5)$$

Our first task is to obtain an explicit expression for the resonant frequency from Eq. A-4. From Eq. 59 we may eliminate X_I and obtain

$$\frac{4m\pi}{N\left(\frac{\omega_{om} d}{c}\right)} + \frac{1}{i} \frac{a^* - a}{(1+a)(1+a^*)} = 0 \quad (A-6)$$

We know that

$$a = \frac{Z_O - R_t}{Z_O + R_t} \exp\left(-\frac{2i\omega L}{c}\right) \quad (A-7)$$

and if we let $Z_O/R_t = q \gg 1$, we have

$$\begin{aligned} \frac{a^* - a}{(1+a)(1+a^*)} &= \frac{2i \sin\left(\frac{2\omega L}{c}\right) \left(\frac{q-1}{q+1}\right)}{1 + \left(\frac{q-1}{q+1}\right)^2 + 2\left(\frac{q-1}{q+1}\right) \cos \frac{2\omega L}{c}} \\ &= \frac{(q^2 - 1) \tan\left(\frac{\omega L}{c}\right)}{q^2 + \tan^2\left(\frac{\omega L}{c}\right)} \quad (A-8) \end{aligned}$$

Accordingly the resonant frequency is given by

$$\frac{4m\pi}{N\left(\frac{\omega_{om} d}{c}\right)} + \frac{(q^2 - 1) \tan\left(\frac{\omega_{om} L}{c}\right)}{q^2 + \tan^2\left(\frac{\omega_{om} L}{c}\right)} = 0 \quad (A-9)$$

It is usually the case that

$$q^2 \gg \frac{4m\pi}{N\left(\frac{\omega_{om} d}{c}\right)} \quad (A-10)$$

in which case the resonant frequency is given by

$$\tan\left(\frac{\omega_{om}L}{c}\right) \approx -\frac{4m\pi}{N\left(\frac{\omega_o d}{c}\right)} \quad (A-11)$$

Now $4m\pi/N(\omega_o d/c) \gg 1$ so that the lowest resonance angular frequency is given by

$$\frac{\omega_{om}L}{c} \approx \frac{\pi}{2} \quad (A-12)$$

an approximation that is accurate enough for our present purposes.

To find

$$R_I = Z_o \frac{(1 - \alpha\alpha^*)}{(1 + \alpha)(1 + \alpha^*)} \quad (A-13)$$

in terms of q at resonance we substitute in Eq. A-13 from Eq. A-7 and obtain

$$\begin{aligned} R_I &= Z_o \frac{q \left[1 + \tan^2\left(\frac{\omega_{om}L}{c}\right) \right]}{q^2 + \tan^2\left(\frac{\omega_{om}L}{c}\right)} \\ &\approx \frac{Z_o}{q} \tan^2\left(\frac{\omega_{om}L}{c}\right) \approx \frac{Z_o}{q} \left[\frac{4m\pi}{N\left(\frac{\omega_o d}{c}\right)} \right]^2 \end{aligned} \quad (A-14)$$

since

$$q^2 \gg \left[\frac{4m\pi}{N\left(\frac{\omega_o d}{c}\right)} \right]^2.$$

At resonance, therefore, the left hand side of Eq. A-1, given by Eq. A-5, is equal to

$$\frac{Z_o^2}{R_I} \frac{1}{\left[\frac{N\left(\frac{\omega_{om}d}{c}\right)}{4m\pi} \right]^2} \approx Z_o q \quad (A-15)$$

It remains to express q in terms of the bandwidth and hence of the magnetron Q , Q_m .

In the neighborhood of the resonance frequency the output impedance of the magnetron is effectively that of a single tuned circuit. The frequency at which the square of the absolute value of this impedance is one-half the value at resonance is given, from Eq. A-3, by the equation

$$iX_I \pm iR_I = -\frac{i 4m\pi}{N\left(\frac{\omega_{om}d}{c}\right)} \approx i \tan\left(\frac{\omega_{om}L}{c}\right) \quad (A-16)$$

Let this frequency be written as

$$\nu_{om} \pm \Delta\nu = \nu_{om} \pm \frac{\Delta\omega}{2\pi} \quad (\text{A-17})$$

so that $2\Delta\nu$ is the bandwidth of the output impedance at the half power points.

$$\begin{aligned} X_I &\approx Z_o \tan\left(\frac{\omega L}{c}\right) = Z_o \tan(\omega_{om} \pm \Delta\omega) \frac{L}{c} \\ &\approx Z_o \left[\tan\left(\frac{\omega_{om} L}{c}\right) \pm \tan \frac{\Delta\omega L}{c} \left(1 + \tan^2\left(\frac{\omega_{om} L}{c}\right)\right) \right] \end{aligned} \quad (\text{A-18})$$

if

$$\tan \frac{\Delta\omega L}{c} \cdot \tan\left(\frac{\omega_{om} L}{c}\right) \ll 1$$

so that, as

$$R_I \approx \frac{Z_o}{q} \tan^2\left(\frac{\omega_{om} L}{c}\right) \quad (\text{A-19})$$

we have

$$\frac{Z_o}{q} = Z_o \tan \frac{\Delta\omega L}{c} \approx Z_o \frac{\Delta\omega L}{c} \approx Z_o \frac{2\Delta\omega}{\omega_{om}} \frac{\omega_{om} L}{2c} \quad (\text{A-20})$$

But from Eq. A-12)

$$\frac{\omega_{om} L}{c} \approx \frac{\pi}{2}$$

and

$$\frac{2\Delta\omega}{\omega_{om}} = \frac{2\Delta\nu}{\nu_{om}} = \frac{1}{Q_m} \quad ,$$

Hence

$$\frac{1}{q} = \frac{\pi}{4Q_m} \quad , \quad \text{and} \quad q = \frac{4}{\pi} Q_m = \frac{Z_o}{R_t} \quad (\text{A-21})$$

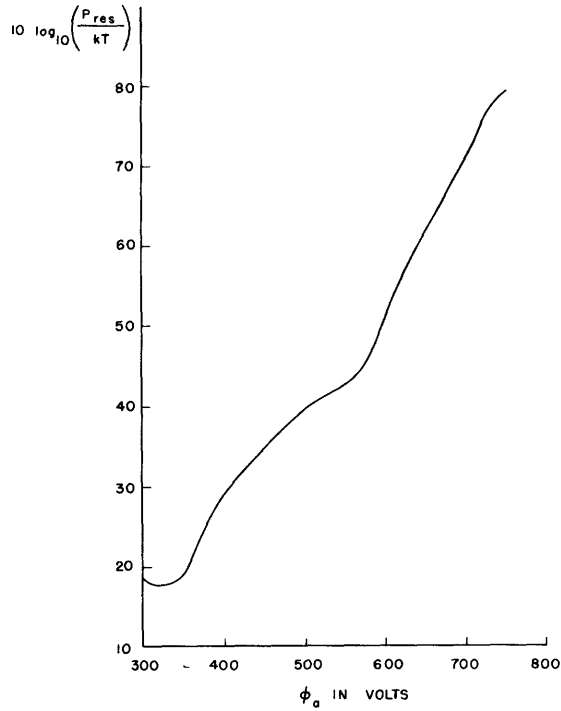
as stated in Eq. 52.

We also have the left hand side of Eq. A-1 identically equal to the right hand side for $\nu \approx \nu_{om}$.

IV. EXPERIMENTAL RESULTS AND POSSIBLE MECHANISMS FOR SPACE CHARGE NOISE AMPLIFICATION

A. Experimental Results and Excess Noise. To check the shot noise theory developed in the first part of this paper it is obviously desirable to measure the noise power produced by a low current density temperature limited magnetron. At the time of writing a suitable tube is under construction but no experimental data are available. Instead we shall use some results, contained in unpublished work by V. Mayper, which

were obtained with a magnetron, Type QK61, operated in a space charge limited, high current density condition, with fixed magnetic field of 0.17 weber/m^2 and variable plate voltage. In Fig. 4 we have plotted the logarithm of the ratio of the observed output noise



power developed at the resonance associated with the π -mode to the thermal noise power generated in the load as a function of ϕ_a , the anode-cathode potential. This curve possesses two very striking features: the enormous noise power generated at the higher plate voltages, and the very rapid increased noise power with plate voltage. Neither of these phenomena can be explained by the shot noise alone, as we shall now show.

Fig. 4 Noise power measured in 16-slot magnetron as function of plate voltage for $B = 0.17 \text{ weber/m}^2$.

From Eq. 56 the shot noise power delivered to an external load is given by

$$\frac{P(\nu)}{kT} = \frac{2eA I_s}{kT} \frac{Nd}{2h} \left(\frac{z}{m}\right)^2 \frac{4Q^2}{\pi Q_L} Z_o \omega^2 \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \quad (64)$$

where

$$|Y_{mc}|^2 = \left| \int_{-\infty}^{\infty} \cos m\theta(r)^m \left[1 - \left(\frac{r_c}{r}\right)^{2m} \right] \exp(-i\omega t) dt \right|^2$$

$$|Y_{ms}|^2 = \left| \int_{-\infty}^{\infty} \sin m\theta(r)^m \left[1 - \left(\frac{r_c}{r}\right)^{2m} \right] \exp(-i\omega t) dt \right|^2 \quad (65)$$

From the data of Fig. 1

$$\begin{aligned} Nd &= 0.006 \text{ m} & r_a &= 2.8 \text{ mm} \\ h &= 0.01 \text{ m} & m &= 8 \\ r_c &= 1.75 \text{ mm} & A &= 1.1 \text{ cm}^2 \end{aligned} \quad (66)$$

The other quantities depend on the external fields.

Let us consider the case where $\phi_a = 750$ volts, $B = 0.17$ weber/m². Under these conditions it was found experimentally that

$$Q = 400 \quad Q_L = 500 \quad (67)$$

The circulating current density I_s was calculated from the steady state magnetron theory of R.L.E. Technical Report No. 118. We have that

$$I_s = 5.1 \cdot 10^4 \text{ amp/m}^2 \quad (68)$$

We also have that

$$\sigma = 5.5 \quad \text{and} \quad K = 3000 \quad (69)$$

where σ and K are parameters of the steady state space charge distribution and that $\ell = 0.295$ mm where $r_c + \ell$ is the radius of the edge of the space charge cloud.

To obtain a numerical expression for the shot noise power it is necessary to evaluate $|Y_{mc}|^2 + |Y_{ms}|^2$. The exact calculation of this quantity, which is determined by the electron's orbit, can only be carried out numerically. However it is easy to obtain an upper limit to this quantity and it is shown in the appendix of Technical Report No. 118 that

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} < \tau^2 \left(\frac{r_c + \ell}{r_a} \right)^{2m} \left[1 - \left(\frac{r_c}{r_c + \ell} \right)^{2m} \right]^2 \quad (70)$$

where τ is the total transit time. Hence

$$\frac{P(\nu_{mo})}{kT_r} < \frac{2eAI_s}{kT_r} \frac{Nd}{2h} \left(\frac{2}{m} \right)^2 \frac{4Q^2}{\pi Q_L} (\omega_{mo}\tau)^2 \left(\frac{r_c + \ell}{r_a} \right)^{2m} \left[1 - \left(\frac{r_c}{r_c + \ell} \right)^{2m} \right]^2 \quad (71)$$

and substituting the numerical values given above and calculating τ , we have

$$10 \log_{10} \left(\frac{P(\nu_{mo})}{kT_r} \right) < 10 \log_{10} 1.27 \cdot 10^6 = 57 \text{ db} \quad (72)$$

where T_r is taken as 290° K.

Now from Fig. 4 we see that the measured noise power produced when the plate voltage is 750 volts is 78 db above the thermal noise which is 21 db greater than the maximum possible shot noise.

If we use the approximate expression for $|Y_{mc}|^2 + |Y_{ms}|^2$ which is given in the appendix of Technical Report No. 118, we have

$$10 \log_{10} \left(\frac{P(\nu_{mo})}{kT_r} \right) \approx 57 - 10 \log_{10} \frac{1}{2} \left(\frac{r_c + 0.35\ell}{r_c + \ell} \right)^{16} \quad (73)$$

$$\approx 48 \text{ db} \quad (74)$$

so that, with the maximum plate voltage we see that the measured noise is about 30 db greater than the theoretical shot noise.

It is even simpler to show that the variation of noise output power with plate voltage is inexplicable on the assumption that only shot noise is present. In Eq. 64 only two of the terms depend on the plate voltage: the dynamic Q and the orbit dependent term

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} .$$

The experimental variation of Q with plate voltage for the particular external fields given above is shown in Fig. 10 of Technical Report No. 118. Over most of the voltage range Q is approximately equal to 300 although it does drop sharply to below 100 when the plate voltage is in the immediate neighborhood of 300 volts.

The change with plate voltage of the orbit dependent term is more complex but, except for voltages so low that the electrons never leave the neighborhood of the cathode, an upper limit to this change will be given by taking the term r^m in Eq. 65 equal to r_c^m . Accordingly the maximum change in shot noise power due to plate voltage is of the order of

$$10 \log_{10} \left(\frac{Q_{\max}}{Q_{\min}} \right)^2 \cdot \left(\frac{r_c + l}{r_c} \right)^{16} = 13 \text{ db}$$

assuming that Q_{\max} is 400 and Q_{\min} is 300.

Since the observed change is of the order of 60 db it is obvious that the experimental data cannot be explained by the shot noise alone.

In Technical Reports No. 117 and 118 we shall try to set up a theory to explain the above discrepancies, which will also throw light on the excess anode current and moding properties of the magnetron.

APPENDIX I.

In this appendix we have to obtain an approximate expression for

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \quad (A-1)$$

where $|Y_{mc}|^2$ and $|Y_{ms}|^2$ are given by Eq. 65 and both r and θ are complex functions of time given by

$$\theta = \int_0^t \dot{\theta} dt = \int_0^t \frac{\omega_0}{2} \left(1 - \frac{r_c^2}{r^2} \right) dt \quad (A-2)$$

$$t = \int_{r_c}^r \frac{dr}{\sqrt{\frac{2kT}{m_o} K \left(\frac{r - r_c}{\ell} \right)^{3/2} \left(1 - \sqrt{\frac{r - r_c}{\ell}} \right)}} \quad (A-3)$$

where $r_1 = r_c + \ell$ is the maximum distance reached by the electron.

Since $|Y_{mc}|^2$ and $|Y_{ms}|^2$ are both proportional to

$$r^{2m} \left[1 - \left(\frac{r_c}{r} \right)^{2m} \right]^2 \quad (A-4)$$

we shall obtain an upper limit to the value of

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}}$$

if we suppose that the electron travels at its maximum radius r_1 for a time equal to τ the total transit time. This suggests that we evaluate

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}}$$

for the case where the electron moves at constant radius r_o for a certain time τ_o . We have in this case

$$\begin{aligned} \frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} &= \left(\frac{r_o}{r_a} \right)^{2m} \left[1 - \left(\frac{r_c}{r_o} \right)^{2m} \right]^2 \left\{ \left| \int_0^{\tau_o} \cos m \dot{\theta}_o t \exp(-i\omega t) dt \right|^2 + \left| \int_0^{\tau_o} \sin m \dot{\theta}_o t \exp(-i\omega t) dt \right|^2 \right\} \\ &= \left(\frac{r_o}{r_a} \right)^{2m} \left[1 - \left(\frac{r_c}{r_o} \right)^{2m} \right]^2 \left\{ \left| \left[\frac{\sin \left(\frac{\omega - m \dot{\theta}_o}{2} \tau_o \right)}{\omega - m \dot{\theta}_o} \exp \left(\frac{im \dot{\theta}_o \tau_o}{2} \right) + \frac{\sin \left(\frac{\omega + m \dot{\theta}_o}{2} \tau_o \right)}{\omega + m \dot{\theta}_o} \exp - \frac{im \dot{\theta}_o \tau_o}{2} \right] \right|^2 \right. \\ &\quad \left. + \left| \left[\frac{\sin \left(\frac{\omega - m \dot{\theta}_o}{2} \tau_o \right)}{\omega - m \dot{\theta}_o} \exp \left(\frac{im \dot{\theta}_o \tau_o}{2} \right) - \frac{\sin \frac{\omega + m \dot{\theta}_o}{2} \tau_o}{\omega - m \dot{\theta}_o} \exp - \frac{im \dot{\theta}_o \tau_o}{2} \right] \right|^2 \right\} \\ &= \left(\frac{r_o}{r_a} \right)^{2m} \left[1 - \left(\frac{r_c}{r_o} \right)^{2m} \right]^2 \frac{1}{2} \left\{ \frac{\sin^2 \left(\frac{\omega - m \dot{\theta}_o}{2} \tau_o \right)}{(\omega - m \dot{\theta}_o)^2} + \frac{\sin^2 \left(\frac{\omega + m \dot{\theta}_o}{2} \tau_o \right)}{(\omega + m \dot{\theta}_o)^2} \right\} \end{aligned} \quad (A-5)$$

For small $\frac{\omega + m\dot{\theta}_0}{2} \tau_0$ this gives us

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} < \left(\frac{r_o}{r_a}\right)^{2m} \left[1 - \left(\frac{r_c}{r_o}\right)^{2m}\right]^2 \tau_0^2 \quad (A-6)$$

an upper limit to

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}}$$

is obtained by taking $r_o = r_1$ and $\tau_o = \tau$ which yields us the inequality in Eq. 67 given in the main body of the text.

For normal values of r_1 , the term

$$\left[1 - \frac{r_c}{r_1}^{2m}\right]^2$$

is nearly equal to one. We might expect therefore to obtain a lower limit to the value of

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}}$$

by taking r_1 equal to r_c and putting $\left[1 - (r_c/r_1)^{2m}\right]^2$ equal to unity.

Perhaps a better approximation would result if we took r_o and θ_o equal to their average values. Now the average value of r is

$$r = r_c + \frac{2}{\tau} \int_0^\ell x \, dt = r_c + 0.35\ell \quad (A-7)$$

so that we may take

$$r_o = r_c + 0.35(r_1 - r_c) \quad (A-8)$$

In this case

$$m\dot{\theta}_0 = \frac{m\omega_o}{2} \left[1 - \left(\frac{r_c}{r_o}\right)^2\right] \approx \frac{m\omega_o}{2} \cdot 0.7 \frac{r_c - r_c}{r_c} \quad (A-9)$$

In the body of the text we are considering the case where

$$\tau_o = \frac{11.3}{\omega_o}, \quad \omega_o = 2.98 \cdot 10^{10}, \quad \omega = 6\pi \cdot 10^9, \quad r_1 - r_c = 0.295 \cdot 10^{-3}$$

so that, in this case,

$$\frac{\omega - m\dot{\theta}_0}{2} \tau_o < 1 \quad \frac{\omega + m\dot{\theta}_0}{2} \tau_o > 2\pi$$

and we have

$$2 \frac{\sin^2\left(\frac{\omega - m\dot{\theta}_0}{2} \tau_0\right)}{(\omega - m\dot{\theta}_0)^2} + \frac{\sin^2\left(\frac{\omega + m\dot{\theta}_0}{2} \tau_0\right)}{(\omega + m\dot{\theta}_0)^2} \approx \frac{\tau^2}{2} \quad (A-10)$$

Approximately therefore

$$\frac{|Y_{mc}|^2 + |Y_{ms}|^2}{r_a^{2m}} \approx \left(\frac{r_l}{r_a}\right)^{2m} \left[1 - \left(\frac{r_c}{r_l}\right)^{2m}\right]^2 \tau^2 \frac{1}{2} \left(\frac{r_c + 0.35l}{r_c + l}\right)^{16} \quad (A-11)$$

This is a reasonable approximation for the particular case considered in the text. For other numerical cases it is necessary to adopt different approximation, or rely on numerical integration.

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